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Reduction of the two-body problem with central interaction on simply connected spaces of constant sectional curvature

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Abstract. The problem of two particles with central interaction on simply connected spaces of a constant curvature was considered. Due to the absence of the Galilei transformation in this case the reduction to the dynamic problem in four-dimensional phase space was carried out using the Marsden–Weinstein method. Canonically conjugate coordinates were found. The classification of reduced dynamic systems was given. For some of them conditions of the existence global solutions for dynamic equations with attractive potentials were found. The comparison of the structure of obtained Hamiltonians with integrals of the one-particle problem with Bertrand's potentials was carried out.

1. Introduction

Among complete homogeneous Riemannian spaces along with the Euclidean space $\boldsymbol{E}^n$, only hyperbolic space $\boldsymbol{H}^n$, the sphere $\boldsymbol{S}^n$ and the projective space $\boldsymbol{S}^n/(\pm I)$, have the greatest $n(n+1)/2$-parametric group of global isometries [1]. Such spaces are homogeneous and isotropic, i.e. they have no chosen points and directions, and they therefore have a constant sectional curvature. They appear for $n = 3$ as space-like sections of a spacetime in some general relativity models. This fact makes the study of various mechanical systems on these spaces interesting.

In the 19th century J Bertrand set up and solved the problem of the determination of all potentials in the Euclidean space, in which any finite motion of a particle was closed. Therefore this problem and its generalizations carry his name. Its solutions are the Coulomb and oscillator potentials. In [2] it was shown that among natural one-particle mechanical systems with a central potential on $\boldsymbol{S}^n$, i.e. systems with a Lagrange function of the form:

$$L = \frac{m}{2} g_{ij}\dot{x}^i\dot{x}^j - U(\rho(x, x_0))$$

where g_{ij} is the metric tensor, and $\rho(x, x_0)$ is the distance between points x and x_0, corresponding to this metric, there are two systems, with all finite trajectories closed. Potentials of these systems are analogues of the Coulomb and the oscillator ones in the space E^n and pass into them when converging a radius of a curvature to infinity.

The Coulomb potential is the fundamental solution for the Laplace–Beltrami operator, due to that it was known before [2], and the quantum-mechanical problem for it was considered in [3, 4], and also later in [5]. These potentials for the space $\boldsymbol{S}^n$ (as solutions of the Bertrand problem) were rediscovered in [6, 7], and for the space $\boldsymbol{H}^n$ they were

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found in [8]. In [2, 9] a discrete spectrum is obtained for corresponding quantum-mechanical problems in the space $\boldsymbol{S}^n$ by means of an algebra of operators, commuting with a Hamiltonian, and being analogues of integrals of classical motions. In [10] quantum-mechanical problems with the Coulomb potential on $\boldsymbol{H}^n$ and $\boldsymbol{S}^n$ were studied by a method of functional integrals. In [8] Kepler laws were generalized as classical problems with the Coulomb potential, in [11] a one-particle integrable classical problem on spaces $\boldsymbol{H}^n$ and $\boldsymbol{S}^n$ with a potential, being a sum of $2(n+1)$ oscillator potentials, was investigated.

In this paper a two-particle classical problem is studied on spaces $\boldsymbol{H}^n$ and $\boldsymbol{S}^n$, which in contrast to the Euclidean case, is not reduced to a one-particle problem, because of an absence for these spaces of analogues of the Galilei transformation, permitting us in the Euclidean case to pass to the coordinate system, connected with the centre of mass. In fact by using a method of the group analysis of differential equations [12], it is possible to show directly, that the group of point transformations, preserving the equation for the geodesic on spaces $\boldsymbol{H}^2$ and $\boldsymbol{S}^2$, is generated only by isometries of these spaces, dilatations, shifts and reflection of time. In other words it is possible to state the following. We shall define by analogy with the Galilei spacetime [14], the spacetime $\mathcal{E}_M$, as a trivial bundle $\pi : \mathcal{E}_M = M \times \boldsymbol{E}^1 \to \boldsymbol{E}^1$, with an axis of time as a base, naturally lift geodesics from M on $\mathcal{E}_M$ and define the group of automorphisms $\mathrm{Aut}(\mathcal{E}_M)$, as a set of diffeomorphisms $\mathcal{E}_M$ on itself, passing a lay in a lay, geodesic in geodesic and inducing an isometry in the base $\boldsymbol{E}^1$. In the case $M = \boldsymbol{E}^2$ $\dim\mathrm{Aut}(\mathcal{E}_M) = 6$, and in a case $M = \boldsymbol{H}^2$ or $\boldsymbol{S}^2$ $\dim\mathrm{Aut}(\mathcal{E}_M) = 4$. The matter is similar in spaces $\boldsymbol{H}^n$ and $\boldsymbol{S}^n$. Thus we can see, that already a two-particle problem on simply connected spaces of constant sectional curvatures can be difficult to solve, while for the Euclidean space difficulties arise from the three-particle case. We shall note that for spaces $\boldsymbol{H}^n$ and $\boldsymbol{S}^n, n \geqslant 3$ the two-particle problem reaches the maximal generality at $n = 3$, i.e. two elements from the space $T^*\boldsymbol{H}^n$ (from the space $T^*\boldsymbol{S}^n$) are always contained in some subspace $T^*\boldsymbol{H}^3 \subset T^*\boldsymbol{H}^n$ (in a subspace $T^*\boldsymbol{S}^3 \subset T^*\boldsymbol{S}^n$). Therefore two mass points with central interaction will always stay in the space $\boldsymbol{H}^3$ ($\boldsymbol{S}^3$). We therefore consider $n \leqslant 3$.

We shall consider a system of two particles, interacting by a central potential. We shall denote by digit 1 a particle with a mass m_1, and by digit 2 a particle with a mass m_2. For separating the motion of a system as a whole we shall use the Marsden–Weinstein method of a reduction of Hamiltonian systems with symmetries [15]. As an alternative to the Galilei transformation while separating the motion of the centre of mass this method was applied to the Euclidean space in [17]. The Abelian subgroup of a complete isometry group of $\boldsymbol{E}^n$, consisting of translations and being isomorphic $\boldsymbol{R}^n$ was used there as a group of symmetries. Further reasonings, using the complete group of isometries, in a limit of a zero curvature, give for the space $\boldsymbol{E}^n$ another way of transforming of a two-particle problem to a one-particle problem. In [16] it was shown that at the reduction of Hamilton dynamic systems on the cotangent bundle of the configuration space by means of some symplectomorphic group, the obtained dynamic system can be identified with the dynamic system on some subbundle of the cotangent bundle of some subspace of the initial configuration space. The basic difficulty at the analysis of a concrete problem is the choice of convenient canonically conjugate coordinates and their interpretation.

2. Basic notations

In order to reduce the consequent calculations we chose models of the spaces $\boldsymbol{H}^3$ and $\boldsymbol{S}^3$ so that by formal replacements it would be possible to transform statements, valid for the one space, into statements valid for the other. Let the sphere $\boldsymbol{S}^3$ be realized as $\boldsymbol{R}^3 \cup \{\infty\}$

with the metric:

$$ds^2 = \frac{4R^2(dx^2 + dy^2 + dz^2)}{(1 + x^2 + y^2 + z^2)^2} \tag{1}$$

where R is the radius of a curvature. In the given model Euclidean angles coincide with non-Euclidean ones. The distance between two points we shall denote as $\rho^s(\cdot\,,\cdot)$.

The component of the unit of the isometry group is the group SO(4), with Lie algebra so(4) = so(3) ⊕ so(3). The resulting Killing vector fields correspond to basis elements of this algebra:

$$\begin{aligned}
X_1^s &= y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \qquad & Y_1^s &= \frac{1}{2}(1 + x^2 - y^2 - z^2)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z} \\
X_2^s &= z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \qquad & Y_2^s &= \frac{1}{2}(1 + y^2 - x^2 - z^2)\frac{\partial}{\partial y} + xy\frac{\partial}{\partial x} + yz\frac{\partial}{\partial z} \\
X_3^s &= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \qquad & Y_3^s &= \frac{1}{2}(1 + z^2 - x^2 - y^2)\frac{\partial}{\partial z} + xz\frac{\partial}{\partial x} + yz\frac{\partial}{\partial y}.
\end{aligned}$$

Commutative relations of these fields have the form:

$$[X_i^s, X_j^s] = -\sum_{k=1}^{3}\varepsilon_{ijk}X_k^s \qquad [Y_i^s, Y_j^s] = -\sum_{k=1}^{3}\varepsilon_{ijk}X_k^s \qquad [X_i^s, Y_j^s] = -\sum_{k=1}^{3}\varepsilon_{ijk}Y_k^s$$

where ε_{ijk} is a completely antisymmetric tensor, $\varepsilon_{ijk} = 1$. If we denote $L_i = \frac{1}{2}(X_i^s + Y_i^s)$, $G_i = \frac{1}{2}(X_i^s - Y_i^s)$, $i = 1, 2, 3$, we shall receive the commutative relations

$$[L_i^s, L_j^s] = -\sum_{k=1}^{3}\varepsilon_{ijk}L_k^s \qquad [G_i^s, G_j^s] = -\sum_{k=1}^{3}\varepsilon_{ijk}G_k^s \qquad [L_i^s, G_j^s] = 0$$

$i, j = 1, 2, 3$, which correspond to the expansion so(4) = so(3) ⊕ so(3).

For the space $\boldsymbol{H}^3$ we take the Poincaré model in the unit ball $D^3 \subset \boldsymbol{R}^3$ with the metric:

$$ds^2 = \frac{4R^2(dx^2 + dy^2 + z^2)}{(1 - x^2 - y^2 - z^2)^2} \qquad x^2 + y^2 + z^2 < 1.$$

The component of the unit of the isometry group is the group SO(1, 3), with the simple algebra so(1, 3). Again Euclidean angles coincide with the non-Euclidean ones. The distance between the two points we shall denote as $\rho^h(\cdot,\cdot)$.

Corresponding Killing vector fields have the form:

$$\begin{aligned}
X_1^h &= y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \qquad & Y_1^h &= \frac{1}{2}(1 - x^2 + y^2 + z^2)\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y} - xz\frac{\partial}{\partial z} \\
X_2^h &= z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \qquad & Y_2^h &= \frac{1}{2}(1 - y^2 + x^2 + z^2)\frac{\partial}{\partial y} - xy\frac{\partial}{\partial x} - yz\frac{\partial}{\partial z} \\
X_3^h &= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \qquad & Y_3^h &= \frac{1}{2}(1 - z^2 + x^2 + y^2)\frac{\partial}{\partial z} - xz\frac{\partial}{\partial x} - yz\frac{\partial}{\partial y}.
\end{aligned}$$

Commutative relations of these fields have the form:

$$[X_i^h, X_j^h] = -\sum_{k=1}^{3}\varepsilon_{ijk}X_k^h \qquad [Y_i^h, Y_j^h] = \sum_{k=1}^{3}\varepsilon_{ijk}X_k^h \qquad [X_i^h, Y_j^h] = -\sum_{k=1}^{3}\varepsilon_{ijk}Y_k^h.$$

In appendix A invariants $F_1^{s,h}$ and $F_2^{s,h}$ are given for coadjoint actions of isometry groups of spaces $\boldsymbol{H}^3$ and $\boldsymbol{S}^3$. Also a classification of the corresponding orbits is given.

3. Reduction of the dynamic system on the space $\boldsymbol{S}^3$

We shall now consider the phase space

$$M = \mathrm{T}^*(\boldsymbol{S}^3 \oplus \boldsymbol{S}^3 \setminus \mathrm{diag})$$

with the diagonal in the configuration space excluded to avoid difficulties with the collisions of two particles. This space possesses the standard symplectic structure: $\omega = \mathrm{d}\boldsymbol{p}\wedge\mathrm{d}\boldsymbol{q}$, where $\boldsymbol{q}$ are coordinates on $\boldsymbol{S}^3$, and $\boldsymbol{p}$ are corresponding impulses. The action of the isometry group on the configuration space naturally rises up to the action on the cotangent bundle, and it is identical for each of the two terms.

The momentum map

$$\Phi : M \to \mathrm{so}^*(4)$$

is given by the formulae:

$$\sum_{i=1}^{2}(y_i p_{z_i} - z_i p_{y_i}) = \phi_1 \qquad \sum_{i=1}^{2}(z_i p_{x_i} - x_i p_{z_i}) = \phi_2 \qquad \sum_{i=1}^{2}(x_i p_{y_i} - y_i p_{x_i}) = \phi_3$$

$$\sum_{i=1}^{2}(\tfrac{1}{2}(1 + x_i^2 - y_i^2 - z_i^2)p_{x_i} + x_i y_i p_{y_i} + x_i z_i p_{z_i}) = \phi_4$$

$$\sum_{i=1}^{2}(\tfrac{1}{2}(1 + y_i^2 - x_i^2 - z_i^2)p_{y_i} + x_i y_i p_{x_i} + y_i z_i p_{z_i}) = \phi_5$$

$$\sum_{i=1}^{2}(\tfrac{1}{2}(1 + z_i^2 - x_i^2 - y_i^2)p_{z_i} + x_i z_i p_{x_i} + y_i z_i p_{y_i}) = \phi_6.$$

The rank of the map Φ is investigated in appendix B. It is shown there, that those and only those values of Φ are regular, for which $F_1^s \neq F_2^s$. Values of Φ, for which $F_1^s = F_2^s$ are irregular and correspond to the motion of particles on $\boldsymbol{S}^2 \subset \boldsymbol{S}^3$. The rank of the map Φ is less than 6 on those points of space M, which correspond to the motion of particles on a common geodesic.

Since the action of the symmetry group SO(4) on M corresponds to its coadjoint action on $\mathrm{so}^*(4)$ (thus the momentum map is intertwining one for two representations), it is possible to choose on each orbit of this group some point so that calculations will be simplified.

We shall consider a value of the momentum map, laying on an orbit of type I (see appendix A). Let $\phi_1 = \alpha$, $\phi_4 = \beta$, $\phi_2 = \phi_3 = \phi_5 = \phi_6 = 0$. Without losing generality it is possible to consider $\alpha, \beta \geqslant 0$, by replacing, if it is necessary, the signs of some coordinates. Initially we shall consider the case $\alpha, \beta > 0$, $\alpha \neq \beta$. The given value of the momentum map is regular and the set $M_{\alpha,\beta} := \Phi^{-1}(\alpha, 0, 0, \beta, 0, 0)$ is a smooth manifold. The stationary subgroup is $G_{st} = \boldsymbol{S}^1 \oplus \boldsymbol{S}^1$, generated by vector fields X_1^s and Y_1^s and acts freely on $M_{\alpha,\beta}$. Corresponding to the common scheme, the phase space of the reduced mechanical system will be the four-dimensional quotient space $\hat{M}_{\alpha,\beta} = G_{st} \setminus M_{\alpha,\beta}$. Let $\pi : M_{\alpha,\beta} \to \hat{M}_{\alpha,\beta}$ be the natural projection, X, Y the vector fields on $\hat{M}_{\alpha,\beta}$,

$$\hat{\omega}(X, Y) = \omega|_{M_{\alpha,\beta}}(\pi^{-1}(X), \pi^{-1}(Y))$$

—the symplectic structure on $\hat{M}_{\alpha,\beta}$. The last definition is correct, since it does not depend on a choice of elements from sets $\pi^{-1}(X)$ and $\pi^{-1}(Y)$ [15]. Next we shall derive convenient coordinates on the reduced phase space $\hat{M}_{\alpha,\beta}$.

Each orbit of the group G_{st} on $\boldsymbol{S}^3$ contains only one point of the form $x = z = 0$, $y \geqslant 0$, therefore it is possible to identify the space $\hat{M}_{\alpha,\beta}$ with a submanifold $M_{\alpha,\beta}$, given by equations

$$\sum_{i=1}^{2}(y_i p_{z_i} - z_i p_{y_i}) = \alpha \tag{2}$$

$$\sum_{i=1}^{2}(z_i p_{x_i} - x_i p_{z_i}) = 0 \tag{3}$$

$$\sum_{i=1}^{2}(x_i p_{y_i} - y_i p_{x_i}) = 0 \tag{4}$$

$$\sum_{i=1}^{2}(\tfrac{1}{2}(1 + x_i^2 - y_i^2 - z_i^2) p_{x_i} + x_i y_i p_{y_i} + x_i z_i p_{z_i}) = \beta \tag{5}$$

$$\sum_{i=1}^{2}(\tfrac{1}{2}(1 + y_i^2 - x_i^2 - z_i^2) p_{y_i} + x_i y_i p_{x_i} + y_i z_i p_{z_i}) = 0 \tag{6}$$

$$\sum_{i=1}^{2}(\tfrac{1}{2}(1 + z_i^2 - x_i^2 - y_i^2) p_{z_i} + x_i z_i p_{x_i} + y_i z_i p_{y_i}) = 0 \tag{7}$$

and the condition, that the point C, located on the geodesic Γ, connecting points 1 and 2, so, that $\rho^s(2, C) = \mu\rho^s(2, 1)$, where μ is some number, has coordinates $(0, y, 0)$, $\quad y \geqslant 0$, see figure 1.

Geodesics Γ and $(0C)$ are located in some Euclidean plane $\hat{\boldsymbol{E}}^2$, which together with the point at infinity is a sphere $\boldsymbol{S}^2$, with respect to the metric (1). Let $(0t)$ be a Euclidean line, perpendicular to the line $(0y)$, located in $\hat{\boldsymbol{E}}^2$ and being geodesic in $\boldsymbol{S}^3$. Let ϕ and ψ be angles between geodesics, as shown in figure 1. Thus

$$r_1 = \tan(\rho^s(1, C)/2R) = \tan((1 - \mu)\arctan r) \tag{8}$$

$$r_2 = \tan(\rho^s(2, C)/2R) = \tan(\mu \arctan r) \tag{9}$$

$$r = \tan(\rho^s(1, 2)/2R). \tag{10}$$

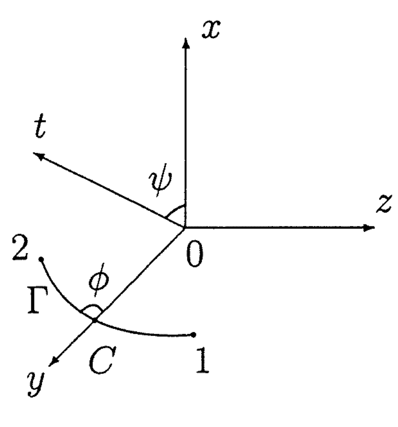

Figure 1. Parametrization of the two-particle system on the space $\boldsymbol{S}^3$.

Using standard formulae of the spherical geometry, it is possible to obtain the following relations:

$$t_1 = -\frac{r_1(1+y^2)\sin\phi}{1+r_1^2y^2-2yr_1\cos\phi} \qquad t_2 = \frac{r_2(1+y^2)\sin\phi}{1+r_2^2y^2+2yr_2\cos\phi}$$

$$y_1 = \frac{y(1-r_1^2)-r_1(y^2-1)\cos\phi}{1+r_1^2y^2-2yr_1\cos\phi} \tag{11}$$

$$y_2 = \frac{y(1-r_2^2)+r_2(y^2-1)\cos\phi}{1+r_2^2y^2+2yr_2\cos\phi}. \tag{12}$$

Besides, it is obvious, that

$$x_1 = t_1\cos\psi \qquad x_2 = t_2\cos\psi \tag{13}$$

$$z_1 = -t_1\sin\psi \qquad z_2 = -t_2\sin\psi. \tag{14}$$

We shall substitute values $x_1, x_2, y_1, y_2, z_1, z_2$, expressed via r_1, r_2, y, ϕ, ψ, in equations (2)–(7) and consider them as a system of linear equations with respect to impulses with a matrix P. Since further evaluations are very cumbersome, they were done using the system of computer analytical transformations Maple V. We shall describe the results obtained. The matrix P has a rank of 5 and the condition of solvability of the system (2)–(7) looks like:

$$\tan\psi = \frac{\alpha(1-y^2)}{2\beta y}. \tag{15}$$

If this condition is satisfied the solution of the system (2)–(7) looks like:

$$\boldsymbol{p} = \boldsymbol{p}^{(0)} + \boldsymbol{u}v \tag{16}$$

where $\boldsymbol{p}^{(0)}$ is a particular solution of the system (2)–(7), $\boldsymbol{u} \in \ker P, \boldsymbol{u} \neq 0, v \in \mathbb{R}^1$. It is possible to choose $\boldsymbol{p}^{(0)}$ and $\boldsymbol{u}$, such that, in view of relations (15) and (8)–(10), the symplectic structure on the space $\hat{M}_{\alpha,\beta}$ ($\omega = d\boldsymbol{p}\wedge\boldsymbol{q}$, where $\boldsymbol{p}$ is given by formula (16), and $\boldsymbol{q}$ by formulas (11)–(14)) looks like:

$$\omega = \mathrm{d}\nu\wedge\mathrm{d}r + \mathrm{d}\left(\frac{\alpha\beta}{\sqrt{\alpha^2\cos^2\psi+\beta^2\sin^2\psi}}\right)\wedge\mathrm{d}\phi.$$

Introducing a notation $p_r = \nu, \quad p_\phi = \alpha\beta/\sqrt{\alpha^2\cos^2\psi+\beta^2\sin^2\psi}$ we obtain, with the help of Maple V, for the kinetic energy of the system, equal to

$$T_s = \sum_{i=1}^{2}\frac{(1+x_i^2+y_i^2+z_i^2)^2(p_{x_i}^2+p_{y_i}^2+p_{z_i}^2)}{8m_iR^2}$$

an expression, depending on μ.

There are two basic different possibilities of the choice of the value μ. The first consists of fixing the centre of mass of the system on the geodesic $(0y)$ by means of the group G_{st} action. In this case $\mu = m_1/(m_1+m_2)$. The second possibility consists of fixing on the geodesic $(0y)$ the positions of one of particles, for example, particle 2. Here $\mu = 0$. In the first case the expression for the kinetic energy takes the form:

$$T_s = \frac{(1+r^2)^2p_r^2}{8mR^2} + \frac{1}{2R^2}A(r)p_\phi^2 + \frac{\alpha^2+\beta^2}{2(m_1+m_2)R^2} + \frac{\alpha^2\beta^2}{2R^2p_\phi^2}C(r)$$
$$+\frac{\sqrt{(\alpha^2-p_\phi^2)(p_\phi^2-\beta^2)}}{4R^2}B(r)\cos\phi + \frac{(\alpha^2-p_\phi^2)(p_\phi^2-\beta^2)}{2R^2p_\phi^2}C(r)\cos^2\phi \tag{17}$$

where

$$A(r) = \frac{(1+r^2)^2}{8mr^2} - \frac{1}{m_1+m_2} + \frac{1-r^4}{8mr^2}\cos\zeta + \frac{1+r^2}{4m_1m_2r}(m_1-m_2)\sin\zeta$$
$$B(r) = \frac{(m_2-m_1)}{m_1m_2r}(1+r^2)\cos\zeta + \frac{1-r^4}{2mr^2}\sin\zeta$$
$$C(r) = \frac{(1+r^2)^2}{8mr^2} - \frac{1}{m_1+m_2} - \frac{1-r^4}{8mr^2}\cos\zeta - \frac{1+r^2}{4m_1m_2r}(m_1-m_2)\sin\zeta$$
$$\zeta = 2\frac{m_1-m_2}{m_1+m_2}\arctan r \qquad m = \frac{m_1m_2}{m_1+m_2}$$
$$\min\{\alpha,\beta\} \leqslant p_\phi = \alpha\beta/\sqrt{\alpha^2\cos^2\psi + \beta^2\sin^2\psi} \leqslant \max\{\alpha,\beta\}.$$

At $\mu = 0$ the expression for the kinetic energy looks like:

$$T_s = \frac{(1+r^2)^2}{8mR^2}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) - \frac{p_\phi^2}{m_2R^2} + \frac{\alpha^2+\beta^2}{2m_2R^2} - \frac{\sqrt{(\alpha^2-p_\phi^2)(p_\phi^2-\beta^2)}}{2R^2m_2}\left(\frac{1+r^2}{p_\phi}p_r\sin\phi + \frac{1-r^2}{r}\cos\phi\right). \tag{18}$$

Expression (17) is symmetric with respect to the permutation of particles, but expression (18) is simpler.

We shall now consider the case $\alpha = \beta \neq 0$. This is the regular value of the momentum map with the stationary subgroup $\boldsymbol{S}^1 \oplus \mathrm{SO}(3)$ generated by vector fields H_1, G_1, G_2, G_3. In this case the obtaining canonical coordinates and the expression the kinetic energy via them can be carried out according to the same scheme with significant simplifications, however, the same result is obtained by passing to the limit in formula (17) or formula (18). The kinetic energy in this case looks like:

$$T_s^{(1)} = \frac{(1+r^2)^2}{8mR^2}\left(p_r^2 + \frac{\alpha^2}{r^2}\right)$$

corresponding to the motion of one particle with a mass m in a central field with constant impulse $p_\phi = \alpha$.

The case $\alpha = 0$, $\beta \neq 0$ or $\alpha \neq 0$, $\beta = 0$, as above-mentioned, corresponds to the motion of particles on $\boldsymbol{S}^2 \subset \boldsymbol{S}^3$, and this value of the momentum map is irregular. However, (see appendix B), the momentum map for two particles on the sphere $\boldsymbol{S}^2$ is regular at all nonzero values and the corresponding stationary subgroup is $\boldsymbol{S}^1$. It is again possible to obtain canonical coordinates and to express the kinetic energy via them using the same scheme with significant simplifications, however the same result can be obtained by passing to the corresponding limit in formulae (17) and (18).

The case of the zero value of the momentum map corresponds to one-dimensional motion and is thus uninteresting.

4. Reduction of the dynamic system on the space $\boldsymbol{H}^3$

Since the formal change of variables $x \to \mathrm{i}x$, $y \to \mathrm{i}y$, $z \to \mathrm{i}z$, $r \to \mathrm{i}r$, $R \to \mathrm{i}R$, $X_k^s \to X_k^h$, $Y_k^s \to Y_k^h$, $k = 1, 2, 3$, $\boldsymbol{p} \to -\mathrm{i}\boldsymbol{p}$, $\phi_k \to \phi_k$, $k = 1, 2, 3$, $\phi_k \to -\mathrm{i}\phi_k$, $k = 4, 5, 6$ transforms formulae, valid for $\boldsymbol{S}^3$, into formulae, valid for $\boldsymbol{H}^3$, we can easily obtain results for $\boldsymbol{H}^3$ from the results for $\boldsymbol{S}^3$.

If the value of the momentum map lays on an orbit of coadjoint action of the group SO(1,3) of the type I (see appendix A), the expression for the kinetic energy for the centre of mass fixed on the axis $(0y)$ by the action of the stationary subgroup is obtained from (17) by the above-mentioned change of variables and looks like:

$$\begin{aligned} T_h = {} & \frac{(1-r^2)^2 p_r^2}{8mR^2} + \frac{1}{2R^2}A(r)p_\phi^2 + \frac{\beta^2-\alpha^2}{2(m_1+m_2)R^2} - \frac{\alpha^2\beta^2}{2R^2 p_\phi^2}C(r) \\ & + \frac{\sqrt{(p_\phi^2-\alpha^2)(p_\phi^2+\beta^2)}}{4R^2}B(r)\cos\phi + \frac{(\alpha^2-p_\phi^2)(p_\phi^2+\beta^2)}{2R^2p_\phi^2}C(r)\cos^2\phi \end{aligned} \tag{19}$$

where

$$\begin{aligned} A(r) &= \frac{(1-r^2)^2}{8mr^2} + \frac{1}{m_1+m_2} + \frac{1-r^4}{8mr^2}\cosh\zeta - \frac{1-r^2}{4m_1m_2r}(m_1-m_2)\sinh\zeta \\ B(r) &= \frac{(m_2-m_1)}{m_1m_2r}(1-r^2)\cosh\zeta + \frac{1-r^4}{2mr^2}\sinh\zeta \\ C(r) &= \frac{(1-r^2)^2}{8mr^2} + \frac{1}{m_1+m_2} - \frac{1-r^4}{8mr^2}\cosh\zeta + \frac{1-r^2}{4m_1m_2r}(m_1-m_2)\sinh\zeta \\ \zeta &= 2\frac{m_1-m_2}{m_1+m_2}\operatorname{arctanh} r \end{aligned}$$

$$\alpha \leqslant p_\phi = \alpha\beta/\sqrt{\beta^2\sin^2\psi - \alpha^2\cos^2\psi} < \infty.$$

At $\mu = 0$ the expression for the kinetic energy looks like:

$$\begin{aligned} T_h = {} & \frac{(1-r^2)^2}{8mR^2}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) + \frac{p_\phi^2}{R^2m_2} + \frac{\beta^2-\alpha^2}{2m_2R^2} \\ & - \frac{\sqrt{(p_\phi^2-\alpha^2)(p_\phi^2+\beta^2)}}{2R^2m_2}\left(\frac{1-r^2}{p_\phi}p_r\sin\phi + \frac{1+r^2}{r}\cos\phi\right). \end{aligned} \tag{20}$$

In the case of belonging of the value of the momentum map to orbits of the coadjoint action of the group SO(1,3), corresponding to the two-dimensional motion of two particles on the space $\boldsymbol{H}^2 \subset \boldsymbol{H}^3$, an expression for kinetic energies can be, as well as for the sphere $\boldsymbol{S}^3$, obtained by limiting invariants of the coadjoint action (or α and β) to corresponding values. Each case of the two-dimensional motion can be called, according to the type of one-dimensional stationary subgroup of the group SO(1,2) (see appendix A), elliptic, hyperbolic and parabolic. Expressions for kinetic energies in these cases are obtained by assuming in the formulae (19) and (20) that $\beta = 0$ (elliptic case), $\alpha = 0$ (hyperbolic case), $\alpha = 0$, $\beta = 0$ (parabolic case).

Remark. From the limiting procedure it is not initially clear what the possible values of the impulse p_ϕ are. Detailed calculations for the sphere $\boldsymbol{S}^2$ and the space $\boldsymbol{H}^2$, which are technically much easier than for the sphere $\boldsymbol{S}^3$ and the space $\boldsymbol{H}^3$, show, that for the sphere $\boldsymbol{S}^2$ there is fulfilled inequality $|p_\phi| \leqslant |\alpha|$ (at $\beta = 0$), and for the space $\boldsymbol{H}^2$ there are fulfilled inequalities as follows. For the elliptic case, $|p_\phi| \geqslant |\alpha|$, for the parabolic case, $|p_\phi| > 0$. For the hyperbolic case the impulse p_ϕ can be of any value.

5. Discussion

Reduced Hamiltonians of particles, interacting by a central potential, are obtained from expressions for kinetic energies (17)–(20) by adding a potential $V(r)$. Hamiltonians corresponding to expressions (18) and (20) after discarding constant terms have the form:

$$H_s = H_s^0 - \frac{p_\phi^2}{m_2 R^2} - \frac{\sqrt{(\alpha^2 - p_\phi^2)(p_\phi^2 - \beta^2)}}{2R^2 m_2} \left(\frac{1+r^2}{p_\phi} p_r \sin\phi + \frac{1-r^2}{r} \cos\phi \right) \tag{21}$$

where

$$H_s^0 = \frac{(1+r^2)^2}{8mR^2} \left(p_r^2 + \frac{p_\phi^2}{r^2} \right) + V(r)$$

and

$$H_h = H_h^0 + \frac{p_\phi^2}{R^2 m_2} - \frac{\sqrt{(p_\phi^2 - \alpha^2)(p_\phi^2 + \beta^2)}}{2R^2 m_2} \left(\frac{1-r^2}{p_\phi} p_r \sin\phi + \frac{1+r^2}{r} \cos\phi \right) \tag{22}$$

where

$$H_h^0 = \frac{(1-r^2)^2}{8mR^2} (p_r^2 + \frac{p_\phi^2}{r^2}) + V(r).$$

Their structure can be described as follows. The first term corresponds to a motion of a particle with a mass m on $\boldsymbol{S}^3$ or $\boldsymbol{H}^3$ in the potential $V(r)$. The second term is an integral for the Hamiltonian $H_{s,h}^0$. Thus, if we omit in Hamiltonians considered the third terms, we shall obtain Hamiltonians, corresponding to integrable systems. Of course, when $V(r) \equiv 0$ Hamiltonians $H_{s,h}$ are also integrable, because they correspond to an independent geodesic motion of two particles. The additional integral in this case is the sum of the second and the third terms.

The closure of all limited orbits in the Coulomb or the oscillator potential for one-particle problems on two-dimensional surfaces of a constant curvature leads to the existence of an additional third integral with respect to the energy and the angular momentum. For the Hamiltonian H_h^0 with the Coulomb potential: $V_h^q(r) = -k(1+r^2)/(2Rr)$ it is possible to use any of the following expressions as such integrals:

$$I_h^{(1)} = ((1+r^2)p_\phi^2/(2Rr) - km)\cos\phi + (1-r^2)p_r p_\phi \sin\phi/(2R)$$
$$I_h^{(2)} = ((1+r^2)p_\phi^2/(2Rr) - km)\sin\phi - (1-r^2)p_r p_\phi \cos\phi/(2R)$$

and for the Hamiltonian H_s^0 with the potential: $V_s^q(r) = -k(1-r^2)/(2Rr)$ from the following [2]:

$$I_s^{(1)} = ((1-r^2)p_\phi^2/(2Rr) - km)\cos\phi + (1+r^2)p_r p_\phi \sin\phi/(2R)$$
$$I_s^{(2)} = ((1-r^2)p_\phi^2/(2Rr) - km)\sin\phi - (1+r^2)p_r p_\phi \cos\phi/(2R).$$

These integrals are the analogues of the components of the Laplace vector for the Euclidean case. We note that the last terms in (21) and (22) are similar to the expression $I^{(1)}$, this likeness is particularly noticeable for the Hamiltonian (22) at $\alpha = \beta = 0$, which corresponds to the parabolic case. It can be rewritten as:

$$H_h^p = K - \frac{km}{m_2 R}\cos\phi$$

where the Hamiltonian

$$K = H_h^0 + \frac{p_\phi^2}{m_2 R^2} - \frac{I_h^{(1)}}{m_2 R}$$

corresponds to an integrable system with the additional integral H_h^0 or $p_\phi^2/(m_2 R^2) - I_h^{(1)}/(m_2 R)$.

At a small value of the parameter $m/m_2 = m_1/(m_1+m_2)$, which corresponds to the case of motion of a light body around a heavy one, Hamiltonians (21) and (22) can be considered as perturbations of integrable one-particle problems. For a potential $V(r)$ of a general form (under the condition of nondegeneracy or the condition of isoenergetic nondegeneracy of a Hamiltonian), the KAM theorem [13] suggests that the majority of invariant tori in a phase space at a sufficiently small parameter m/m_2 are only slightly deformed, and action variables (constructed for nonperturbed problems) will always remain near their initial values. In other words, for finite orbits the time of motion between pericentre and apocentre, and also the distance from these points to the point $r = 0$ will always remain near their initial values.

However, for the Coulomb and the oscillatory potentials the condition of the nondegeneracy is not fulfilled and the KAM theorem cannot be applied.

A proof of the impossibility of the collision of particles is of some interest. It is clear, that for repulsion potentials with a singularity in the point $r = 0$ collisions do not happen.

Theorem. If the potential $V(r)$ is smooth at $r > 0$ and has at $r \to 0$ a singularity $\mathrm{o}(r^{-2})$, collisions of particles for the finite time will not happen in the following cases.

(1) For the space $\boldsymbol{S}^3$ at $F_1^s \neq 0$, $F_2^s \neq 0$; and

(2) for the space $\boldsymbol{H}^3$ at $F_1^h \neq 0$, $F_2^h \neq 0$ and also in elliptic and parabolic cases of motion on the space $\boldsymbol{H}^2 \subset \boldsymbol{H}^3$. If these conditions are fulfilled, then the dynamic system of two particles on the sphere $\boldsymbol{S}^3$ has a global solution. If, in addition, the potential $V(r)$ is bounded below, at $r \to 1$, this is also valid for the space $\boldsymbol{H}^3$. Particularly, conditions of the theorem are satisfied by potentials, being solutions of the Bertrand problem.

Proof. Let a potential $V(r)$ be smooth at $r > 0$ and $V(r) = \mathrm{o}(r^{-2})$ at $R \to 0$. In formulae (17) and (19) functions A, B, C have the following asymptotics at $r \to 0$:

$$A(r) = \frac{1}{4mr^2} + \mathrm{O}(1) \qquad B(r) = \mathrm{O}(r) \qquad C(r) = \mathrm{O}(r^2). \tag{23}$$

Therefore in the Hamiltonian $H_s := T_s + V(r)$, where T_s is given by formula (17) there can be singular, at $r \to 0$ only the terms $(1+r^2)^2 p_r^2/(8mR^2)$, $A(r)p_\phi^2/(2R^2)$, $V(r)$, since $|p_\phi| \geqslant \min\{|\alpha|, |\beta|\} > 0$. However, it is clear, that their sum $\to \infty$ at $r \to 0$, that contradicts the energy preservation law: $H = \text{constant}$.

It is also valid for the Hamiltonian, corresponding to the kinetic energy (19) at $\alpha \neq 0$. It is only necessary to carry out the proof of the collisions impossibility in the parabolic case, i.e. for the Hamiltonian

$$H_h = \frac{(1-r^2)^2}{8mR^2} p_r^2 + \left(A(r) + \frac{1}{2} B(r) \cos(\phi) - C(r) \cos^2(\phi) \right) \frac{p_\phi^2}{2R^2}. \tag{24}$$

We shall carry out this proof by *reductio ad absurdum*, assuming, that at time τ the value $r = 0$. Then from formulae (23) and (24) and the energy preservation law, it follows that $p_\phi(\tau) = 0$. However, the value p_ϕ satisfies the following differential equation

$$\frac{\mathrm{d}p_\phi}{\mathrm{d}t} = -\frac{\partial H_h}{\partial \phi} = \frac{p_\phi^2}{2} \left(\frac{1}{2} B(r) \sin(\phi) - C(r) \sin 2\phi \right).$$

One can see at $\tau_1 < \tau$ and sufficiently small value $\tau - \tau_1$ that:

$$|p_\phi(\tau_1)| = \left| \int_\tau^{\tau_1} \frac{\mathrm{d}p_\phi}{\mathrm{d}t} \mathrm{d}t \right| \leqslant \text{constant} \int_\tau^{\tau_1} |p_\phi| \mathrm{d}t.$$

From Gronwall's inequality we obtain:

$$|p_\phi(\tau_1)| \leqslant |p_\phi(\tau)| \exp(\text{constant}|\tau - \tau_1|) = 0.$$

However, the condition $p_\phi \equiv 0$ contradicts the remark from section 4.

The drift of a particle (particles) at finite time to infinity can also prevent the existence of a global solution of corresponding dynamic systems on the space $\boldsymbol{H}^3$. If the potential is bounded from below at $r \to 1$ for $\boldsymbol{H}^3$, it does not happen because of the energy preservation low. $\square$

Appendix A

The orbits classification of the coadjoint action of isometry groups of spaces $\boldsymbol{S}^3$ and $\boldsymbol{H}^3$.

Since SO(4) $\simeq$ (SU(2) $\oplus$ SU(2))/$\{\pm(1 \oplus 1)\}$, and SO(3) $\simeq$ SU(2)/$\{\pm 1\}$, the orbit of the coadjoint action of the group SO(4) is a direct product of two orbits of the coadjoint action of the group SO(3). The latter orbits are concentric spheres $\boldsymbol{S}^2 \subset \mathbb{R}^3$, or $0 \in \mathbb{R}^3$. We shall denote the basis in so$^*(4)$, dual to the basis $X_i^s, Y_i^s, \quad i = 1, 2, 3$ by X_s^i, Y_s^i, so, that an arbitrary element from so$^*(4)$ looks like $\sum_{i=1}^3 (\phi_i X_s^i + \phi_{i+3} Y_s^i)$. Then an orbit of the coadjoint action of the group SO(4) is given by equations

$$F_1^s := \sum_{i=1}^3 (\phi_i + \phi_{i+3})^2 = c_1$$

$$F_2^s := \sum_{i=1}^3 (\phi_i - \phi_{i+3})^2 = c_2 \qquad c_i \geqslant 0 \qquad i = 1, 2.$$

Each orbit contains a point of the form $\phi_1 = \alpha, \phi_4 = \beta, \phi_2 = \phi_3 = \phi_5 = \phi_6 = 0$. At $\alpha \neq \pm\beta$ inequalities $c_1 \neq 0, c_2 \neq 0$ are fulfilled, the orbits are isomorphic to the space $\boldsymbol{S}^2 \oplus \boldsymbol{S}^2$, the stationary subgroup is $\boldsymbol{S}^1 \oplus \boldsymbol{S}^1$ and generated by vectors X_1^s, Y_1^s. We shall label such orbits as orbits of type I. At $\alpha = \pm\beta \neq 0$ either $c_1 = 0$, or $c_2 = 0$, orbits are isomorphic to the space $\boldsymbol{S}^2$, the stationary subgroup is $\boldsymbol{S}^1 \oplus$ SO(3). We shall label such orbits as orbits of type II. At $\alpha = \beta = 0$ the orbit consists only of the point 0, and the stationary subgroup is SO(4).

We shall denote the basis in so$^*(1, 3)$, dual to the basis X_i^h, Y_i^h, $i = 1, 2, 3$ by X_h^i, Y_h^i, such that an arbitrary element from so$^*(1, 3)$ looks like $\sum_{i=1}^3 (\phi_i X_h^i + \phi_{i+3} Y_h^i)$. Then an orbit of the coadjoint action of the group SO$(1, 3)$ is given by equations

$$F_1^h := \sum_{i=1}^3 (\phi_i^2 - \phi_{i+3}^2) = c_1$$

$$F_2^h := \sum_{i=1}^3 (\phi_i^2 \phi_{i+3}^2) - 2\phi_1\phi_4\phi_2\phi_5 - 2\phi_1\phi_4\phi_3\phi_6 - 2\phi_2\phi_5\phi_3\phi_6 = c_2.$$

These orbits can be classified as follows. Orbits, for which $c_2 \neq 0$, contain a point of the form $\phi_1 = \alpha \neq 0$, $\phi_4 = \beta \neq 0$, $\phi_2 = \phi_3 = \phi_5 = \phi_6 = 0$. The corresponding stationary subgroup is generated by vectors X_1^h, Y_1^h and is equal to $\boldsymbol{S}^1 \oplus \boldsymbol{R}^1$. We shall refer to such orbits as type I. The type II orbits consist of orbits, for which $c_2 = 0, c_1 > 0$. Such orbits contain a point of the form $\phi_1 = \alpha \neq 0, \phi_2 = \phi_3 = \phi_4 = \phi_5 = \phi_6 = 0$. The vector

X_1^h generates one-parameter group of transformations, preserving the stratification $\boldsymbol{H}^3$ on stratums, isomorphic $\boldsymbol{H}^2$, and in each stratum this transformation is elliptic [18]. We shall call, therefore, the given type of orbits elliptic. The corresponding subgroup is generated by vectors X_1^h, Y_1^h. The type III orbits consist of orbits, for which inequalities $c_2 = 0, c_1 < 0$ are fulfilled. Such orbits contain a point of the form $\phi_4 = \beta \neq 0, \phi_1 = \phi_2 = \phi_3 = \phi_5 = \phi_6 = 0$. The vector Y_1^h corresponds to hyperbolic transformations of the space $\boldsymbol{H}^2 \subset \boldsymbol{H}^3$, therefore we shall call the type III orbits hyperbolic. The corresponding stationary subgroup is generated by vectors X_1^h, Y_1^h. The type IV orbits are represented by a unique orbit, which contains the point $\phi_1 = -\phi_6 = 1$, $\phi_2 = \phi_3 = \phi_4 = \phi_5 = 0$. In this case $c_1 = c_2 = 0$. The corresponding stationary subgroup is generated by vectors $X_1^h + Y_3^h$, $X_3^h - Y_1^h$. Vectors X_1^h, Y_2^h, Y_3^h generate the Lie algebra $\mathrm{so}(1,2) \subset \mathrm{so}(1,3)$, and the vector $X_1^h + Y_3^h$ corresponds to parabolic transformations of space $\boldsymbol{H}^2 \subset \boldsymbol{H}^3$. We shall call this orbit parabolic.

All previously mentioned orbits are four-dimensional. Besides, there is only a trivial orbit, consisting only of the point $0 \in \mathrm{so}^*(1,3)$.

Appendix B

The investigation of the momentum map's rank.

We shall consider the momentum map, corresponding to a system of two particles on the space $\boldsymbol{S}^3$. Since the group SO(4) acts by diffeomorphisms on M, and also on $\mathrm{so}^*(4)$, the rank of the momentum map is constant on each orbit and can be calculated on a point with coordinates $x_1 = y_1 = z_1 = y_2 = z_2 = p_{z_2} = 0$, $x_2 \neq 0$, since such point is contained on any orbit of the groups SO(4) on the space M. Let the coordinates on M are ordered as : $x_1, y_1, z_1, x_2, y_2, z_2, p_{x_1}, p_{y_1}, p_{z_1}, p_{x_2}, p_{y_2}, p_{z_2}$. Then the Jacobi matrix of the momentum map has the following block form:

$$\begin{pmatrix} A & B & 0 & C \\ 0 & D & F & Q \end{pmatrix} \tag{25}$$

where

$$A = \begin{pmatrix} 0 & p_{z_1} & -p_{y_1} \\ -p_{z_1} & 0 & p_{x_1} \\ p_{y_1} & -p_{x_1} & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & -p_{y_2} \\ 0 & 0 & p_{x_2} \\ p_{y_2} & -p_{x_2} & 0 \end{pmatrix} \tag{26}$$

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -x_2 \\ 0 & x_2 & 0 \end{pmatrix} \qquad D = x_2 \begin{pmatrix} p_{x_2} & p_{y_2} & 0 \\ -p_{y_2} & p_{x_2} & 0 \\ 0 & 0 & p_{x_2} \end{pmatrix} \tag{27}$$

$$F = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad Q = \frac{1}{2} \begin{pmatrix} 1 + x_2^2 & 0 & 0 \\ 0 & 1 - x_2^2 & 0 \\ 0 & 0 & 1 - x_2^2 \end{pmatrix}. \tag{28}$$

By using a block column $\begin{pmatrix} 0 \\ F \end{pmatrix}$ the Jacobi matrix can be transformed, not changing its rank, to the matrix

$$\begin{pmatrix} A & B & 0 & C \\ 0 & 0 & F & 0 \end{pmatrix} \tag{29}$$

the rank of which is greater by three than the rank of the matrix (ABC). Since $x_2 \neq 0$, the last rank is 3 under the condition $p_{y_1}^2 + p_{z_1}^2 + p_{y_2}^2 \neq 0$ and 2 in the opposite case. The value of the momentum map in the given point is

$$\Phi = (0, 0, x_2 p_{y_2}, \tfrac{1}{2} p_{x_1} + \tfrac{1}{2}(1 + x_2^2) p_{x_2}, \tfrac{1}{2} p_{y_1} + \tfrac{1}{2}(1 - x_2^2) p_{y_2}, \tfrac{1}{2} p_{z_1}) \tag{30}$$

and values of invariants of the co-adjoint action of the group SO(4) are:

$$F_1^s = \tfrac{1}{4}(p_{x_1} + (1+x_2^2)p_{x_2})^2 + \tfrac{1}{4}(p_{y_1} + (1-x_2^2)p_{y_2})^2 + (\tfrac{1}{2}p_{z_1} + x_2 p_{y_2})^2 \tag{31}$$

$$F_2^s = \tfrac{1}{4}(p_{x_1} + (1+x_2^2)p_{x_2})^2 + \tfrac{1}{4}(p_{y_1} + (1-x_2^2)p_{y_2})^2 + (\tfrac{1}{2}p_{z_1} - x_2 p_{y_2})^2. \tag{32}$$

If $p_{y_1} = p_{z_1} = p_{y_2} = 0$, then $F_1^s = F_2^s$. If $F_1^s = F_2^s$, then $x_2 p_{z_1} p_{y_2} = 0$, and either $p_{z_1} = 0$, or $p_{y_2} = 0$. In the first case both particles always remain on the sphere $\boldsymbol{S}^2 \subset \boldsymbol{S}^3$, given by the equation $z = 0$, and in the second case—on the sphere $\boldsymbol{S}^2 \subset \boldsymbol{S}^3$, containing the geodesic $(0x)$ and the geodesic, generated by a vector of a velocity of the first particle. If, in addition, a rank of momentum map is less than 6 (in this case it is equal to 5), in the above chosen point we have: $p_{y_1} = p_{z_1} = p_{y_2} = p_{z_2} = 0$ and the motion of two particles takes place on the geodesic $(0x)$.

Reasoning similarly for the space $\boldsymbol{H}^3$, we obtain the following.

Proposition 1. A value of the momentum map for a system of two particles on the space $\boldsymbol{S}^3$ (on the space $\boldsymbol{H}^3$) is irregular in points, which correspond to particles moving on the sphere $\boldsymbol{S}^2 \subset \boldsymbol{S}^3$ (on the space $\boldsymbol{H}^2 \subset \boldsymbol{H}^3$) and are characterized by the condition $F_1^s = F_2^s \quad (F_2^h = 0)$. The rank of the momentum map is not maximal in those points of the phase space, which correspond to the motion of particles on a common geodesic.

The following statement can be similarly proved.

Proposition 2. A non-zero value of the momentum map for a system of two particles on spaces $\boldsymbol{S}^2$ or $\boldsymbol{H}^2$ is regular. The zero value of the momentum map corresponds to the one-dimensional motion of two particles on a common geodesic with the total zero impulse.

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